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THE APPLICATION OF THE FINITE-ELEMENT METHOD TO METEOROLOGICAL SIMULATIONS—A REVIEW

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SUMMARY

The application of the finite-element method to the simulation of meteorological fluid flow problems is reviewed. Early studies were aimed primarily at demonstrating the viability of the method for one- and two-dimensional flows, whereas more recent studies have been aimed at demonstrating the efficiency and viability of the method for more complex three-dimensional simulations. There has also been a shift towards exploiting such models to better understand and predict the underlying meteorological phenomena, rather than restricting attention to the development of the algorithms.

KEY WORDS Review Finite Elements Meteorology

INTRODUCTION

It is now ten years since the first journal article appeared on the application of the finite-element (FE) method to meteorologically-oriented fluid flow simulations. Although there are several articles of a general nature on this subject,[1–4] none is both complete and up-to-date. The present article is an attempt to review these first ten years of progress and is arranged by area of application and within areas, in chronological order. The author has arbitrarily limited himself to reviewing articles whose prime application is to meteorological simulations; articles in allied fields such as free surface flows and convection problems were deemed to be worthy of separate reviews (by someone else). References to reports, theses and conference proceedings have only been included where the material is not available in a journal article.

BAROTROPIC MODELS

The first application of the FE method to a meteorological flow was that of Wang *et al.*[5] They used cubic Hermite functions with a uniform periodic spacing in x to solve the pair of one-dimensional (1-D) gravity wave equations

$$u_t + uu_x + gh_x = 0 \quad (1)$$

$$h_t + uh_x + hu_x = 0 \quad (2)$$

where u is the fluid velocity in the x direction, h is the depth of the fluid and g is the gravitational acceleration; the linearized forms of (1) and (2) were also examined. Several methods of time discretization were used, all based on the Crank–Nicolson method. Equations (1) and (2) and their linearized forms were integrated and the results compared

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with those of a 4th-order finite-difference (FD) formulation using the same time-discretization methods. It was concluded that for a given accuracy their FE method was computationally more efficient than the corresponding FD method, although the programming was far more complicated for FEs (because of the choice of basis functions and time integration scheme). The generalization of their work to 2- and 3-dimensions, as well as to non-uniform grids would be complicated and was not attempted.

Cullen[6] used bilinear FEs for the 2-D advection equation

$$\phi_t + u\phi_x + v\phi_y = 0 \tag{3}$$

and for the free-surface gravity wave equations

$$u_t + uu_x + vu_y + \phi_x = 0 \tag{4}$$

$$v_t + uv_x + vv_y + \phi_y = 0 \tag{5}$$

$$\phi_t + (u\phi)_x + (v\phi)_y = 0 \tag{6}$$

where u and v are the x and y components of velocity, respectively, and ϕ is the free-surface height. Equation (3) was integrated on a uniform rectangular grid using a leap-frog time integration scheme. An FE integration of (3) for the passive advection of a cone was compared with corresponding 2nd and 4th order accurate FD integrations. A similar comparison was made where the initial conditions were chosen to produce a sharp gradient as a simple model of frontogenesis. A linear stability analysis of (3) showed that the FE scheme required a timestep smaller by a factor of $1/\sqrt{3}$ than that of the 2nd order centered FD scheme. The non-linear equations (4)–(6) were solved on a uniform periodic domain with both leap-frog and Runge–Kutta time schemes and results compared with those of a 2nd order energy-conserving FD scheme. A linear analysis of the phase truncation errors of (4)–(6) was performed for both FD and FE versions and the superiority of the FE scheme, particularly for the smaller space scales, was evident. It was concluded from the above experiments that the FE schemes outperformed the 2nd order FD schemes and were as good as, or better than, 4th order FD methods for equivalent computational cost. The conclusions would presumably have been even more favourable toward the FE method had the more efficient direct method described by Staniforth and Mitchell[7] been used for solving algebraic equations of the form

$$\mathbf{Mf} = \mathbf{r} \tag{7}$$

where $\mathbf{M}$ is the mass-matrix and $\mathbf{f}$ and $\mathbf{r}$ are coefficient vectors, rather than the iterative method used. The previously mentioned timestep penalty of the FE schemes when compared to the 2nd order FD schemes is due to the superior treatment of the shorter space scales; this is not surprising, since a similar result holds when comparing 4th order FD schemes to 2nd order FD schemes. In fact the FE scheme for the linearized problems is 4th order accurate as can be easily seen from an expansion of the phase truncation factor of expression (16) of Cullen's article for small Δx; i.e.

$$\frac{3}{(2+\cos k\Delta x)}\,\frac{\sin k\Delta x}{\Delta x} = \frac{1 - \dfrac{(k\Delta x)^2}{6} + \dfrac{(k\Delta x)^4}{120} + \dots}{1 - \dfrac{(k\Delta x)^2}{6} + \dfrac{(k\Delta x)^4}{72} + \dots} = 1 + O(\Delta x^4) \tag{8}$$

where k is the wavenumber and Δx is the grid-length.

Cullen[8] used linear equilateral triangles and a leap-frog time scheme to integrate the shallow-water equations

$$u_t + uu_x + vu_y + \phi_x - fv = 0 \tag{9}$$

$$v_t + uv_x + vv_y + \phi_y + fu = 0 \tag{10}$$

$$\phi_t + (u\phi)_x + (v\phi)_y = 0 \tag{11}$$

where u, v and ϕ are as before and f is the Coriolis parameter. The equations were solved for a periodic channel on a 'betaplane', i.e.

$$f = f_0 + \beta y \tag{12}$$

where f_0 and β are constant. These equations are perhaps more typical of those of interest in meteorology than the gravity-wave equations of the previous studies, since they include the important effect of rotation and consequently permit non-stationary Rossby waves. Results of these FE integrations were compared with those of various 2nd and 4th order FD schemes and it was concluded that the FE schemes were superior. It was found however that modifying the FE calculation of the non-linear terms by smoothing improved the results, and also that the results were very sensitive to how the boundary conditions were imposed; the conclusions are therefore not clear-cut and subject to interpretation.

Cullen[9], Hinsman[10] and Hinsman and Archer[11] used linear equilateral triangles defined on an icosahedral mesh to solve the shallow-water equations on a sphere for Rossby–Haurwitz waves. Cullen[9] used a leap-frog time scheme whereas Hinsman[10] and Hinsman and Archer[11] used an extrapolated Crank–Nicolson. Cullen[9] reported noise and computational stability problems, particularly near the vertices of the icosahedron and resorted to artificial smoothing as a control mechanism. The results were not entirely satisfactory, although it was concluded that the FE method was reasonably competitive with the spectral and FD comparison models. Hinsman[10] and Hinsman and Archer[11] also experienced some stability difficulties for their longer-time integrations.

Motivated by his previous work, Cullen[12] examined various choices of introducing artificial smoothing to control noise in shallow-water equation models, both for FEs and FDs. However the most interesting aspect of the article is the analysis of the spatial evolutionary error for various schemes; these ideas are further developed by Cullen[3] and Cullen and Morton.[13] It was concluded that the spatial evolutionary error for bilinear elements is $O(h^4)$ on a uniform rectangular mesh of mesh-length h. This important result explains why linear elements (low-order polynomials) on a uniform mesh can compete so successfully with 4th order FD methods. It was also concluded that it is more accurate to compute non-linear terms involving a derivative by first evaluating the derivative and then computing the product, rather than by directly evaluating the product. Higher-order spline bases were also examined and again gave more accurate error estimates than would at first glance be expected (super-convergence at the nodes).

Staniforth and Mitchell[7] examined the efficiency of the FE method for solving the shallow-water equations for a rotating fluid on a polar-stereographic projection of the sphere (this introduces the minor complication of map-scale factors; the algorithms may be straightforwardly applied to plane geometry by setting the map-scale factors to unity). They showed how to implement a semi-implicit time discretization which is approximately four times more efficient than competing methods (such as the leap-frog scheme used by Cullen[6,8,9]), a significant improvement which can be crucial in real-time, synoptic-scale forecasting problems. In so doing, they performed a stability analysis to show that it is advantageous to use vorticity and divergence as predictive variables instead of velocity components; this also resulted in a consistent application of the boundary conditions, a difficulty previously encountered by Cullen.[8] A 4th order accurate method was given for the solution of the elliptic boundary-value problems, and this and the analysis of Cullen[3,12] and Cullen and Morton[13] gives a 4th order estimate for the spatial evolutionary error; this 4th order accurate method for approximating the solution of equations having second derivatives is essential to raising the spatial evolutionary error from 2nd to 4th order, a point apparently overlooked by Cullen (Reference 3, p. 328). The accuracy of their model was demonstrated practically using 500 mb data in comparison experiments with 2nd and 4th order FD models, and it was concluded that the FE method was indeed competitive with the FD method when both use a semi-implicit time discretization. Contrary to the experience of Cullen,[6,8,9,12] there were no noise or stability problems and the model was integrated to 50 days without difficulty.

The generalization of the above work to a variable-resolution Cartesian mesh is given by Staniforth and Mitchell.[14] A method was given for approximating the elliptic boundary-value problems which is 4th order accurate on any uniform sub-domain. For the variable-coefficient Helmholtz problem, the conjugate-gradient method was employed to accelerate the convergence of the iterative procedure. It was shown how to exploit the separability of the basis to efficiently solve equations involving the mass matrix (c.f. (7)). The scheme is optimal, taking $O(MN)$ operations on a $M \times N$ mesh compared with $O(M^2N)$ for banded matrix solvers. It was shown how to evaluate product (non-linear) terms using a product-Simpson quadrature which is more efficient than the usual product-Gaussian rule, both integration rules being exact. The methods employed have the added advantage of being particularly economical for storage. A series of experiments was performed using several mesh configurations, each having uniform high resolution over a specified area of interest and lower resolution elsewhere, to produce short-term forecasts over the high-resolution subdomain without the necessity of high-resolution everywhere. It was found that the forecast produced on a uniform high-resolution mesh can be essentially reproduced for a limited time over the specified area of interest by a variable-mesh configuration at a fraction of the computational cost. Noise problems were avoided by smoothly varying the resolution away from this area. It was consequently concluded that this was a viable strategy for the limited-area/-time numerical weather prediction problem.

Ritchie[15] numerically integrated the non-divergent barotropic vorticity equation on a semi-infinite 'β-plane' to investigate a resonance mechanism for Rossby waves on a shear flow in the presence of a non-linear critical layer. The model employed a Fourier expansion in the periodic (x-) direction and linear FEs in the semi-infinite y-direction; the time-discretization was leap-frog. A weak forcing was assumed at the ($y =$ constant) boundary and the radiation condition of Béland and Warn[16] was applied at the open computational boundary (situated at a finite value of y), with the critical layer lying between these two boundaries. High uniform resolution was used in the y-direction within the critical layer in order to resolve the large local gradients, and the resolution was made to vary smoothly away from this layer to the coarsely-resolved outer region where the gradients are small. It was found that varying the resolution in this way gave results of equivalent accuracy to those of high resolution everywhere, but at half the computational cost. It was shown that a high amplitude response to a weak forcing can be found by suitably varying the parameters of the problem, suggesting that resonance is possible under certain circumstances.

Navon[17] used an extrapolated Crank–Nicholson time scheme with uniform triangular elements to solve the shallow-water equations for a channel on a 'β-plane' (equations (9)–(12)). He experimented with a consistent mass matrix, a lumped mass matrix and a linear combination of the two, and compared these with the results of other authors for the same problem. He concluded that there was a good correspondance between his results and those of Cullen[8] and a 4th order FD model for the same problem. It was found that better results were obtained using a mass matrix defined by the average of the consistent and lumped mass matrices than with either used separately. It was suggested that a judicious linear combination of the two 2nd order accurate methods (consistent and lumped mass matrices) could result in cancellation of the 2nd order error contributions such as to give 4th order accuracy, and that this probably explains the superior performance of the averaged mass matrix. It is interesting to note that the use of bilinear rectangular elements for the same problem automatically gives 4th order accuracy without any modification to the mass matrix; further, contrary to linear triangular elements, any modification of the mass matrix in the latter case will serve to reduce the accuracy rather than increase it. Although he used an extrapolated Crank–Nicolson time scheme to increase the permissible time step, the gain in comparison with a leap-frog scheme seemed to be offset for the most part by the increased number of computations, and thus the semi-implicit time scheme is to be preferred.

Sasaki and Reddy[18] examined the advection of a circular vortex in a periodic channel in the absence of rotation using the 2-D incompressible flow equations. They compared several combinations of space and time schemes using an exact analytical solution as a control. The FD schemes were based on Arakawa's[19] approximation to the Jacobian, whereas the FE schemes used bilinear rectangular elements. (We note in passing that Arakawa's scheme is equivalent to a mass-lumped FE scheme using bilinear square elements as pointed out by Jesperson.[20]) A vorticity/stream function formulation was also compared to that of velocity components. It was concluded that the best results were obtained for the bilinear FEs using vorticity/stream function in conjunction with a Crank–Nicolson time scheme, and that a variational enstrophy adjustment improved the long-time solutions.

Haltiner and Williams[4] formulated a FE model of the barotropic vorticity equation for periodic and contained flows using linear triangular elements in the manner of Fix[21] for an ocean model; no results were given.

Schoenstadt[22] examined the effect of using staggered and unstaggered meshes for the shallow-water equations in their primitive form. Equations (9)–(11) (with $\phi = gh$) were linearized in a 1-D infinite region with no mean flow to give

$$u_t + gh_x - fv = 0 \tag{13}$$

$$v_t + fu = 0 \tag{14}$$

$$h_t + Hu_x = 0 \tag{15}$$

where H is the mean height and u, v and h are the perturbed velocities in the x and y direction and the perturbed free surface height, respectively, f is the Coriolis parameter and g is the acceleration due to gravity. He showed that for both FD and FE formulations an unstaggered arrangement of variables propagates energy in the wrong direction for the smaller scales; this manifests itself in numerical integrations as small-scale noise. Williams[23] extended this work to include unstaggered FD and FE formulations where vorticity and divergence are used as predictive variables instead of velocity components. The linearized equations corresponding to (13)–(15) are then given by

$$\zeta_t + fD = 0 \tag{16}$$

$$D_t + gh_{xx} - f\zeta = 0 \tag{17}$$

$$h_t + HD = 0 \tag{18}$$

where $\zeta = v_x$ is the vorticity and $D = u_x$ is the divergence. An analysis of (16)–(18) for various unstaggered FD and FE formulations showed that such formulations do not suffer the same problems as the unstaggered arrangement formulated in terms of velocity components. The results of Schoenstadt[22] and Williams[23] are important since they explain the noise problems of the models mentioned earlier in this section that use velocity components, and the absence of problems in the vorticity divergence formulations of Staniforth and Mitchell[7,14] for the barotropic problem and of Staniforth and Daley[24] and Cullen and Hall[25] for the baroclinic (3-D) problem.

Williams and Zienkiewicz[26] examined mixed-order elements (linear for velocity components, constant for free-surface height and vice-versa) on a staggered grid for the linearized 1-D shallow-water equations with a mean flow. An analysis similar to that of Schoenstadt[22] indicated that the results should be superior to an unstaggered formulation with linear elements and this was verified. The extensions to non-linear problems and 2-D were not tested and appear difficult.

BAROCLINIC PRIMITIVE EQUATION MODELS

Carson and Cullen[27] compared forecasts made by various versions of an FD multilayer model with those made by a mixed FD/FE model, for two initial data sets for periods of up to 5 days. All models used $\sigma(=p/ps)$ co-ordinates and FD's in the vertical, and a leap-frog time scheme. The FD multi-layer model was solved over a global domain using 2nd order energy-conserving finite differences on a quasi-uniform rectangular grid. The mixed FD/FE model was solved over a hemispherical domain using linear triangular elements in the horizontal in the manner of Cullen[9] for a barotropic model. It was concluded that the forecasts from the mixed FD/FE model were reasonably competitive with those of the other models when verified against real data, but the use of triangles on an icosahedral mesh generated spurious ridging associated with the icosahedral boundaries.

Results from the above mixed FD/FE model for three-day forecasts from an initial data set as well as general circulation simulations were compared in Cullen and Hall[25] with those from a mixed FD/FE model using stream function and velocity potential as dependent variables rather than velocity components. The use of a stream-function/velocity potential (or equivalently, vorticity/divergence) formulation enabled this latter model to employ a semi-implicit time discretization. Comparisons were also made with a spectral and FD model. It was concluded that results from the stream-function/velocity potential version of the mixed FD/FE model was superior to those of the velocity-component version; this result can be explained by the later work of Schoenstadt[22] and Williams.[23] It was also found that results from the spectral model were generally as good as those from the stream-function/velocity potential version of the mixed FD/FE model and the results of both were generally superior to those of the other comparison models.

A finite-element formulation for the vertical discretization of sigma-co-ordinate primitive-equation models was given by Staniforth and Daley.[28] The formulation is independent of the particular choice of horizontal discretization and was tested using a spectral (spherical harmonic) representation in the horizontal and a semi-implicit time discretization. The need to explicitly integrate the hydrostatic equation at each timestep was removed by differentiating the prognostic equation for divergence with respect to σ and using the hydrostatic equation to re-express the (only) term involving the geopotential height in terms of the temperature. A sample 36 h forecast using real data of this fully Galerkin model was given to demonstrate the viability of the method. Daley[29] used the normal modes of this model to develop a procedure for initializing the data of the model, and all but eliminated the unimportant gravity modes while retaining almost unaltered the Rossby modes. The above model has also been used by Daley[30] to examine under what conditions the interaction between geostrophic and ageostrophic modes is a significant feature of predictability decay in deterministic forecast models.

Staniforth and Daley[24] combined the above FE formulation of the vertical discretization with the horizontal FE discretization of Staniforth and Mitchell,[14] to produce an efficient fully FE model using the baroclinic primitive equations. The model has a semi-implicit time scheme and the variable horizontal resolution gives 4th order accurate, noise-free 24 h forecasts for the higher-resolution area of interest situated over N. America. These forecasts are accurate until the coarser outside resolution contaminates the high-resolution inner region of interest, and permit more detailed forecasts in a real-time environment at the expense of a shorter forecast period when compared to models having uniform (but lower) resolution everywhere. Verification scores against analysed observations of both 24 h and 48 h forecasts of this model (54 in all) with those of an operational spectral model were given by Benoit *et al.*,[31] and it was concluded that the variable-resolution FE model was a competitive real-time forecast model for periods of at least 48 h.

MacPherson *et al.*[32] formulated a 3-D primitive equations model in isentropic co-ordinates to model frontogenesis, using bilinear FEs on a rectangular mesh in the horizontal and FDs in the vertical. They compared integrations of this model using two different sets of initial conditions[33,34] with those of an FD model having approximately twice the resolution, and concluded those of the FE model were superior. However, comparisons of these results with those of Williams[33] and Hoskins and Bretherton[34] were not as encouraging. Possible reasons for this include inadequate resolution, difficulty in imposing boundary conditions and the choice of vertical co-ordinate.

The results of Koclas[35] using a 2-D(x-z) FE model for the above problem using the same initial conditions are more convincing. He also used bilinear rectangular elements but employed horizontal mesh-lengths more in keeping with the scales of the physical problem being examined (horizontal mesh-lengths as small as 10 km were used as compared to those of 1000 km of MacPherson *et al.*[32]). Results of various integrations were compared with those of Hoskins and Bretherton[34] and Williams[33] with good agreement; in particular, Koclas[35] obtained slightly steeper gradients across the front at equivalent times than did Williams.[33] The advantage of using variable resolution (high resolution surrounding the front, varying smoothly away from the front) was well demonstrated. Using this approach, results of equivalent quality to those of uniform high-resolution everywhere were produced at half the computational cost.

ATMOSPHERIC BOUNDARY-LAYER AND POLLUTION MODELS

Gresho *et al.*[36] compared results from various linear and quadratic FE approximations to the 1-D and 2-D advection diffusion equation with those from 2nd and 4th order FD schemes. Their paper is a revised version of that presented in June 1976 at the 2nd International Symposium on FE Methods held in Rappallo, Italy. For pure advection in 1-D they concluded that the FE schemes were superior to the FD schemes and that mass lumping seriously reduces accuracy. It was also shown that whereas both quadratic FEs and linear FEs give a 4th order accurate physical mode (the linear elements are a little less accurate, especially for the smallest scales), the quadratic elements give rise to a spurious mode which moves approximately five times as fast as the physical mode and in the wrong direction. In a linear problem this behaviour will not cause any problems provided very little of the initial data is projected on the spurious mode. However, in a non-linear problem the modes are no longer linearly independent and this may cause difficulties. A further drawback to quadratic FEs when compared to linear FEs for this problem is the additional computation time required by the increased band width of the FE matrices. Results of calculations for pure advection in 2-D and for the 2-D transport of a pollutant by the advection–diffusion equation were also reported. The dangers of mass-lumping were again highlighted.

A 1-D atmospheric boundary-layer (ABL) model of moist, deep convection using linear FEs was formulated and tested by Manton.[37] The model predicted the first moments of the dependent variables; closure was achieved by making the covariances of the dependent variables proportional to an eddy diffusivity which depends on the turbulent kinetic energy and a characteristic turbulence time-scale. Results were given for a model problem where condensation, diffusion and shear were included, but Coriolis forces and subsistence were neglected. Experiments were conducted to examine the sensitivity of the results to changes in the variable vertical resolution, timestep, the order of the numerical quadrature and the number of iterations used in the solution procedure; the solutions appeared to have converged and to be physically reasonable for the model problem.

Long *et al.*[38] formulated an ABL model suitable for real-time forecasting of boundary-layer values of wind, temperature and humidity for periods of up to 24 h. Obukhov similarity theory was used for the surface layer and K-theory for the turbulent transfer in the transition layer, while the upper boundary conditions were provided by a prior integration of a large-scale primitive equation model. Numerically, the model used a time-splitting technique, an FD method for vertical diffusion and linear FEs for horizontal advection. A 35×30 horizontal mesh (mesh-length of 80 km) with 10 vertical levels was used but no forecasts were given.

Pepper *et al.*[39] used similar methods to model the dispersion of atmospheric pollution using the 3-D advection–diffusion equation

$$\phi_t + \mathbf{V}\cdot\nabla\phi = \nabla\cdot(\mathbf{K}\nabla\phi) \tag{19}$$

where $\mathbf{V}$ denotes the variable advection wind field, $\mathbf{K}$ the eddy diffusivity and ϕ the concentration. The formulation used similarity- and K-theory to give a cubic profile for the vertical diffusivity and the model was designed to use analysed observations from 7 meteorological towers as initial data. Results using analytically specified initial data for the passive advection of a scalar compared favourably with those of a cubic-spline model at equivalent resolution; no results were given when diffusion was included.

Chan *et al.*[40] formulated two FE models to simulate the spread and dispersion of liquified natural gas (LNG). Both models used the time-dependent, 2-D (x-z), non-divergent conservation equations for mass, momentum and energy and a constant eddy viscosity. One of the models further made the hydrostatic assumption by neglecting the convective motion, inertia, and shear forces in the vertical. Bilinear elements were used for all variables with the exception of the use of constant elements for pressure in the non-hydrostatic model; implicit time-marching schemes were employed. It was concluded from integrations over a 15×24 variable-resolution mesh that both models gave realistic simulations for large diffusivities, but the hydrostatic version was inadequate for several cases of interest. The hydrostatic version also gave somewhat noisy results; this is perhaps not surprising since the use of the hydrostatic assumption as well as the incompressibility assumption probably overconstrains the problem.

The above hydrostatic model has been further developed by Takle *et al.*[41] and Chang *et* *al.*[42] They used a more sophisticated parameterization of turbulence as well as higher-order elements. Takle *et al.*[41] compared results from a 1-D version of the model with other authors and concluded the model gave realistic, accurate simulations. Chang *et al.*[42] discussed preliminary results for the simulation of the sea-breeze.

Chan *et al.*[43] formulated a 3-D FE model for the spread and dispersion of LNG using a modified form of the anelastic equations, and turbulence was parametrized by bulk formulae. Trilinear elements were used for velocity components whereas constant elements were used for pressure. Forward time differences were employed, which is potentially dangerous, since in the absence of diffusion the scheme is numerically unstable. They argued that mass lumping was justifiable on the grounds of cost-effectiveness, even though it degrades accuracy. Results of a simulation by this model of a LNG spill at China Lake, California are given by Chan *et al.*;[44] they appear to be physically reasonable.

An interesting framework for examining the conservation properties of FE formulations of the Boussinesq equations is given by Lee *et al.*[45] They showed how to numerically conserve certain physically conserved quantities in a selective manner by appropriately adding terms to the equations of the form $cF\boldsymbol{\nabla}.\mathbf{V}$, where c is a constant, $\mathbf{V}$ is the 2-D velocity vector and F is one of the dependent variables of the problem. These terms are zero analytically, but numerically of the order of the truncation error, and render the numerical formulation of the problem less 'stiff'. Cliffe[46] corrected an algebraic error in their analysis and in a similar manner to Lee *et al.*,[45] observed that by a suitable choice of parameters one could conserve one of mean temperature (T), mean squared temperature (T^2) and energy (E); with one particular choice it was possible to conserve both T and E for a general choice of elements, but not T^2. However, he observed that if one wishes to conserve all three quantities (T, T^2, E), then it is necessary to use mixed-order interpolation. It was further noted that the element space for pressure elements should not be made too large, otherwise spurious (computational) pressure modes are created.[47] It appears therefore that one should choose the smallest such space to avoid generating computational modes. The above ideas can be used to examine the baroclinic primitive equations by adding weighted multiples of the vertically integrated continuity equation to the governing equations.

Mailhot and Benoit[48] formulated a 1-D ABL suitable for numerical weather prediction models using linear FEs and a Crank–Nicolson time scheme. Their formulation is somewhat similar to that of Manton[37] except that a characteristic length scale instead of a characteristic time scale was used in the eddy diffusivity and they further neglected vertical density variations. They compared their results with other authors and real data and achieved realistic results at much lower resolution than previous studies. They concluded that this should enable the scheme to be used for real-time applications in 3-D.

CONCLUSION

The analyses of Cullen[3,12] and Cullen and Morton[13] give 4th-order error estimates for linear elements on uniform 1-D meshes. This result carries over to 2-D and 3-D for bilinear rectangular and trilinear brick elements, and the accuracy has been verified practically.[6,7,13,14,18,24,39] Given the accuracy, computational efficiency[7,14] and simplicity of these elements, it is usually not cost-effective to use higher-order elements for problems that can be mapped (e.g. by a co-ordinate transformation) to a regular Cartesian mesh. For such problems (and many of the above-discussed problems fall into this category) it is usually less accurate and more costly, for example, to use parabolic elements than linear elements. As indicated by Sani *et al.*,[47] higher-order elements can also generate computational modes (unrelated to the physical solution) and consequently accuracy and stability problems for non-linear time-dependent flows; the imposition of boundary conditions is also more delicate. The increased flexibility of variable resolution for linear, bilinear and trilinear elements when compared to FDs has been found useful by several authors.[14,24,28,32,35,37,40,48]

The above, unfortunately, is not the whole story; one must never lose sight of the properties of the underlying governing equations. This is clearly demonstrated by the work of Schoenstadt[22] and Williams[23] where two different formulations (both using linear elements for all variables) of the same problem give significant differences for the accuracy and stability of the short space-scales. It may therefore be necessary to use mixed-order interpolation, or staggering, or a reformulation of the dependent variables of the problem for best results. The approaches of Schoenstadt[22] and Williams[23] and Lee *et al.*[45] and Cliffe[46] are valuable tools in this regard. Further work in this area, particularly for the vertical formulation of baroclinic primitive equation and Boussinesq models, should shed some more light on the subject.

For problems with truly irregular geometry it appears that there is no choice but to accept the increased computational cost of irregular triangular meshes as the price to be paid for accurate simulations. The increase in computational cost of elements defined on an arbitrary triangulation of a domain is for the most part associated with the solution of algebraic equations of the form (7) involving the mass matrix; the reason is that one can no longer solve such problems as efficiently as for rectangular elements on a regular Cartesian mesh.

Although there are some outstanding computational questions, the application of the FE method to meteorological flows has now passed from the stage of demonstrating its viability to exploiting the models to further understand and predict meteorological phenomena and the future appears promising.

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